NPS55BD71062A

United States Naval Postgraduate School

A SET THEORETIC TREATMENT OF COHERENT SYSTEMS

Richard W. Butterworth

June 1971

Approved for public release; distribution unlimited.

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral A. S. Goodfellow, USN
Superintendent

M. U. Clauser
Provost

ABSTRACT:

This paper develops properties of coherent systems from a set theoretic veiwpoint with particular emphasis on modules of coherent systems. The methods used here demonstrate concisely the previously reported properties of modules as well as some new properties. Specifically, min path sets and min cut sets of a coherent system are given several characterizations. The modularity of a subset of components is given a new characterization which we use as a definition. The situation under which a set and its complement are simultaneously modules is characterized. Finally, the established three modules theorem is given a new proof.

Prepared by:

List of non-standard symbols

$\emptyset$ denotes the empty set

$\forall$ is the quantifier "for all"

$\exists$ is the quantifier "there exists"

$\ni$ means "such that:

ε means "is an element of"

1. Introduction.

Coherent systems arise in the study of reliability when one considers a physical system whose operation is classified as either functioning or failing, and when this operation is determined by the joint functioning of failing of some finite set of components. A "coherent system" is one for which the replacement of a failed component by a functioning one will *not* cause a functioning system to fail.

The basic references which deal with coherent systems, [6] and [8], do so from a boolean function viewpoint. This paper investigates coherent systems and their modules from a set theoretic viewpoint. Modules were originally described in [4]; reference [10] discusses committees of simple games, an equivalent mathematical structure. The methods and some of the results given here are new to the theory of coherent systems. These methods and results provide more tools for the practical use of modules in coherent systems.

We alert the reader to the following notation; $A \subset B$ means A is a *proper* subset of B, and; $A \subseteq B$ means A is simply a subset of B.

2. Coherent Systems.

In reliability contexts, coherent systems are viewed as monotone Boolean functions, that is, binary valued functions of a finite set of binary valued variables. We will use this viewpoint to introduce coherent systems.

Let C be a finite nonempty set (the set of components) and let $\{0,1\}^C$ denote all functions on C to $\{0,1\}$. For $X \in \{0,1\}^C$, we say that X is a *joint performance* of the components C, with the interpretation that: $\forall\, c \in C$,

$$X(c) = \begin{cases} 1 & \text{if component } c \text{ functions} \\ 0 & \text{if component } c \text{ fails.} \end{cases}$$

A system is any function ϕ on $\{0,1\}^C$ to $\{0,1\}$, with the interpretation that, for a joint performance X,

$$\phi(X) = \begin{cases} 1 & \text{if the system functions under the joint performance } X \\ 0 & \text{if the system fails under the joint performance } X. \end{cases}$$

For joint performances X and Y, we say $X \leq Y$ whenever $\forall\ c \in C$, $X(c) \leq Y(c)$. Then (C,ϕ) is a *coherent system* whenever:

$$X \leq Y \Rightarrow \phi(X) \leq \phi(Y). \tag{1}$$

Some components may have no effect on the system's behavior. We classify these as inessential components; all components not inessential will be called essential. Precisely, a component c is *inessential* to (C,ϕ) when, $\forall\ X$, $\phi(0_c,X) = \phi(1_c,X)$, where $(i_c,X)(e) = X(e)$ if $e \neq c$, $= 1$ if $e = c$.

The definition of coherent systems requires, in addition to (1) above,

$$\text{at least one component is essential to } (C,\phi). \tag{2}$$

Examples of coherent systems are the series system on C, for which $\phi(X) = \text{Min}\ \{X(c) \mid c \in C\}$, the parallel system on C, for which $\phi(X) = \text{Max}\ \{X(c) \mid c \in C\}$, and the k-out-of-n system on any n-element set C, for which $\phi(X) = 1 \Leftrightarrow \sum_{c \in C} X(c) \geq k$.

Coherent systems are examined in [4], [6] and [8], while [5] gives an excellent application of coherent systems in formulating a class of life distributions, those with increasing hazard rate average.

2.1 Paths and Cuts.

The following notions are well defined for any function ϕ on $\{0,1\}^C$ to 0,1 . For $A \subseteq C$, let $I_A \varepsilon \{0,1\}^C$ be $= 1$ on A, $= 0$ on C - A. A *path* (*cut*) of (C,ϕ) is any set $P(K) \subseteq C$ such that $\phi(I_P) = 1$ $(\phi(I_{C-K}) = 0)$. A *min path* (*min cut*) is any path (cut) which is set minimal with respect to being a path (cut). That is, P(K) is a path (cut) but no proper subset of it is.

We give a characterization of coherent systems in terms of their min path sets. It is easily proven from the definitions.

Proposition 1:

If (C,ϕ) is a coherent system, then the family of all min paths, $\mathcal{P}$, satisfies:

$$\forall\ P, Q \varepsilon \mathcal{P}, \text{ we have } P \not\subset Q. \tag{3}$$

$$\cup\, \mathcal{P} = \text{the set of essential components of } (C,\phi). \tag{4}$$

Conversely, if $\mathcal{P}$ is a non-empty family of non-empty subsets of C, a finite non-empty set, and if $\mathcal{P}$ satisfies (3), then there exists a uniquely determined coherent system (C,ϕ) which has $\mathcal{P}$ as its family of min paths. It will have $\cup\, \mathcal{P}$ as its set of essential components. Indeed, we can define $\phi(X) = 1 \Leftrightarrow \exists\ P \varepsilon \mathcal{P} \ni X \geq I_P$.

An elementary observation about paths and cuts will be useful in what follows. By virtue of equation (1), knowing all min paths or all min cuts is equivalent to knowing the function ϕ, indeed $\phi(X) = 1$ $\Leftrightarrow \exists$ min path $P \ni X \geq I_P$ or $\phi(X) = 0 \Leftrightarrow \exists$ min cut $K \ni X \leq I_{C-K}$.

The following proposition serves to characterize when two given families of subsets of C are the min path and min cut sets respectively of some coherent system.

Proposition 2.

Let C be a finite nonempty set and let $\mathcal{P}$ and $\mathcal{K}$ be nonempty families of subsets of C. Then $\mathcal{P}$ and $\mathcal{K}$ are the min path and min cut sets respectively of some coherent system with component set C if and only if both $\mathcal{P}$ and $\mathcal{K}$ satisfy:

$$\forall\ P,\ Q \in \mathcal{P},\ P \not\subset Q \quad \text{and} \quad \forall\ K,\ H \in \mathcal{K},\ K \not\subset H \tag{5}$$

and

$$\forall\ A \subseteq C,\ \text{ either } \exists\ P \in \mathcal{P} \ni P \subseteq A \ \text{ or } \ \exists\ K \in \mathcal{K} \ni K \subseteq C - A, \text{ but not both.} \tag{6}$$

Equivalent to (5) and (6) is

$$\forall\ P,\ Q \in \mathcal{P},\quad P \not\subset Q \tag{7}$$

and

$\mathcal{K}$ consists of the set minimal elements of $\{A | A \subseteq C,\ \forall\ P \in \mathcal{P},\ PA \neq \emptyset\}$. (8)

Proof:

Given the coherent system, it is clear from the definitions that equations (5) and (6) are satisfied. Conversely, suppose (5) and (6) are satisfied and define, for $X \in \{0,1\}^C$,

$$\phi(X) = \begin{cases} 1 & \text{if } \exists\ P \in \mathcal{P} \ni X \geq I_P \\ 0 & \text{otherwise.} \end{cases}$$

Now, (C,ϕ) is a coherent system. Because of (5), its min path sets are $\mathcal{P}$. Because of (6), every element of $\mathcal{K}$ is a cut set of (C,ϕ) and $\mathcal{K}$ contains all the min cuts of (C,ϕ). Because of (5), $\mathcal{K}$ contains only min cuts.

To show that (7) and (8) are an equivalent characterization, suppose first that $\mathcal{P}$ and $\mathcal{K}$ are the min paths and min cuts of (C,ϕ). Equation (7) clearly holds. For any subset A of components, if $C - A$ contains no min path, then by (6), A is a cut set. It follows that the sets $\mathcal{K}$ described in (8) are the min cut sets. Conversely, suppose that (7) and (8) hold. Then clearly (5) holds. Let $A \subseteq C$. If A contains no element of $\mathcal{P}$ as a subset, then $C - A$ must intersect every element of $\mathcal{P}$, hence $C - A$ contains an element of $\mathcal{K}$, by (8). If however, A contained a member of $\mathcal{P}$ and $C - A$ a member of $\mathcal{K}$, as subsets, then the given member of $\mathcal{K}$ would not intersect the given member of $\mathcal{P}$, which would be a contradiction. Hence (6) holds.

Remark:

In references [9] and [7], families $\mathcal{P}$ and $\mathcal{K}$ of subsets of C which satisfy equations (5) and (6) are called blocking systems. The equivalence of (7) and (8) to (5) and (6) is also shown, but no identification of a structure function ϕ is useful there. Proposition 2 shows that coherent systems and blocking systems are mathematically equivalent.

2.2 Duality.

The symmetry of equations (5) and (6) make the following definitions and observations straightforward.

Since $I_C \in \{0,1\}^C$ is identically one on C, then for $\phi^d(X) = 1 - \phi(I_C - X)$, where $X \in \{0,1\}^C$, and (C,ϕ) is a coherent system, (C,ϕ^d) is also a coherent system, the *dual of* (C,ϕ). The paths and cuts of (C,ϕ) are the cuts and paths respectively of its dual (C,ϕ^d). This notion of duality is a reasonable one since $(C,(\phi^d)^d) = (C,\phi)$.

These observations imply a dual proposition to Proposition 1, in which we just replace every occurrence of the word "path" by the word "cut" and change the last line to read: "Indeed, we can define $\phi(X) = 0 \Leftrightarrow \exists\, P \in \mathcal{P} \ni X \leq I_{C-P}$."

3. Modules.

Proposition 2 indicates that one can study coherent systems from a Boolean function viewpoint using the structure function ϕ or from a set-theoretic viewpoint using the min path sets $\mathcal{P}$. The former approach is taken in reliability theory and in switching theory while the latter approach is used in combinatorial analysis. This paper uses the set theoretic view of coherent systems to study an important aspect of coherent systems in reliability, namely modules. A module is, both intuitively and mathematically, a subset of components which collectively tends to either aid or deter the system's operation. That is, the influence of the components in a module on the system's operations is either positive or negative, depending on the joint performance of the module's components, but independent of the joint performance of components outside the module.

Modules appear naturally in real systems since such systems are frequently conceived and designed in modular form. They are the natural building blocks for making systems out of subsystems, which are themselves built from other systems, and so on. A practical interest in modules is in computing system reliability as pointed out in [4].

In what follows, the properties of modules are studied from a set theoretic viewpoint. All the properties of modules given in [4] are proved as well as some properties not previously reported.

3.1 Definitions and Equivalences for Modules.

From now on we will assume that our coherent systems have no inessential components, equivalently that the union of all min paths is C. This will avoid needless complications, particularly where modules are concerned.

Let $\mathcal{F}$ be a family of subsets of C, and $A \subseteq C$. The restriction of $\mathcal{F}$ to A, written $\mathcal{F}|_A$, will denote the nonempty intersections of A with the elements of $\mathcal{F}$, that is, $\{FA | F \in \mathcal{F}, FA \neq \emptyset\}$.

Let (C,ϕ) be a coherent system and let A be a nonempty set of components. Letting $\mathcal{P}$ and $\mathcal{K}$ be the min path and min cut sets of (C,ϕ) respectively, we say that A is a *module* of (C,ϕ) whenever $\mathcal{P}|_A$ and $\mathcal{K}|_A$ are the min path and min cut sets respectively of a coherent system with component set A. (Note that we could say, ... with component set C and inessential components C - A, however this leads us away from the above assumption that our coherent systems have no inessential components.)

The usual definition (see below) of a module is equivalent to our definition except for the trivial difference that we accept C as a module. Notice that singleton subsets, i.e., the components themselves, are always modules. For series and parallel systems, every nonempty component set A is a module; for a k-out-of-n system with $1 < k < n$, no subset A except the singletons and C is a module.

The following characterization of modules is useful in the sequel.

Proposition 3.

Let (C,ϕ) be a coherent system with min path sets $\mathcal{P}$. Let $A \subseteq C$ be nonempty. A is a module if and only if

$\forall\ P \in \mathcal{P},\ Q \in \mathcal{P} \ni PA \neq \emptyset$ and $QA \neq \emptyset$ we have that

$$PA \cup Q(C-A) \in \mathcal{P} \tag{9}$$

Proof:

A weaker looking version of equation (9) will be useful here and later.

Lemma: The set A satisfies equation (9) if and only if

$\forall\ P \in \mathcal{P},\ Q \in \mathcal{P} \ni PA \neq \emptyset$ and $QA \neq \emptyset$ we have that

$$PA \cup Q(C-A) \text{ is a path set} \tag{10}$$

Proof of lemma:

Clearly (9) implies (10). Suppose (10) holds but that (9) fails. Then $\exists$ specific min paths P^* and Q^* intersecting A with $P^*A \cup Q^*(C-A) \supset R$, where R is a min path. Either $RA \subset P^*A$ or $R(C-A) \subset Q^*(C-A)$, possibly both. Note first that R intersects A, since if $RA = \emptyset$, then $R \subset Q^*$ which contradicts the minimality of Q^*. Now if $RA \subset P^*A$, then applying equation (10) to R and P^*, $RA \cup P^*(C-A)$ is a path which is strictly contained in P^*, a contradiction. If not, then $R(C-A) \subset Q^*(C-A)$ and applying equation (10) to Q^* and R, $Q^*A \cup R(C-A)$ is a path which is strictly contained in Q^*, a contradiction.

Returning to the main proof, we will show all modules satisfy (9) by showing (10) is satisfied. Letting P and Q be min path sets having a nonempty intersection with A, let $R = PA \cup Q(C-A)$ and suppose R is not a path. Then $C - R$ must be a cut set and hence contain a min cut, say K, i.e., $K \subseteq (C-R)$. It follows that $KA \supseteq KQ \neq \emptyset$ so $KA \in \mathcal{K}|_A$, where $\mathcal{K}$ are all min cuts of (C,ϕ). Since $PA \neq \emptyset$ by assumption, $PA \in \mathcal{P}|_A$. Now, the fact that $(KA)(PA) = \emptyset$ contradicts the modularity of A.

To show that the converse holds, we begin with:

Lemma: If a set A satisfies equation (9), then every element of $\mathcal{P}|_A$ intersects every element of $\mathcal{K}|_A$, i.e., $\forall\ PA \in \mathcal{P}|_A$, $\forall\ KA \in \mathcal{K}|_A$, we have $(PA)(KA) \neq \emptyset$ As usual, $\mathcal{K}$ denotes the min path sets of the given coherent system.

Proof of lemma:

Suppose the lemma is false. Then $\exists\ PA \in \mathcal{P}|_A$ and $KA \in \mathcal{K}|_A$ $\ni (PA)(KA) = \emptyset$. Taking $E = K(C-A)$, it follows, using (5), that for some min path Q, $Q \subseteq C - E$. The inequality $QA \supseteq QKA = QK \neq \emptyset$ allows us to apply the hypothesis, equation (9), to P and Q, giving $PA \cup Q(C-A)$ is a min path. This is a contradiction because $K(PA \cup Q(C-A)) = (PA)(KA) \cup Q\ E = \emptyset$.

Returning to the main proof, we show that A is a module when A satisfies (9), by showing that $\mathcal{P}|_A$ and $\mathcal{K}|_A$ satisfy (5) and (6), relative to the component set A. Suppose $\exists\ PA \in \mathcal{P}|_A$ and $QA \in \mathcal{P}|_A \ni PA \subset QA$. Then $PA \cup Q(C-A)$ is a min path strictly contained in Q, a contradiction. Now suppose $\exists\ KA \in \mathcal{K}|_A$ and $HA \in \mathcal{K}|_A \ni KA \subset HA$ Letting $E = KA \cup H(C-A)$, E can contain no cut hence $\exists$ min path $P \ni P \subseteq (C-E)$. The inequality $PA \supseteq PHA = PH \neq \emptyset$ justifies use of the lemma, concluding $(PA)(KA) \neq \emptyset$ which contradicts the fact $P \subseteq (C-E)$.

Having established equation (5), we show (6) holds. We must show $\forall\ E \subseteq A$, either (a): $\exists\ PA\ \varepsilon\ \mathcal{P}|_A \ni PA \subseteq E$; or (b): $\exists\ KA\ \varepsilon\ \mathcal{K}|_A \ni KA \subseteq A - E$; but not both (a) and (b). Clearly (a) and (b) holding simultaneously would violate the above lemma. Pick any min path $P \ni PA \neq \phi$ and consider $E \cup P(C-A)$. Either this set contains a min path $Q \subseteq (E \cup P(C-A)) \Rightarrow QE \neq \phi \Rightarrow$ (a) holds, or its complement in C, $(A-E) \cup ((C-A)-P) \supseteq K$, a min cut, which $\Rightarrow K(A-E) \neq \phi \Rightarrow$ (b) holds. This completes the proof.

The usefulness of the characterization of modules given in Proposition 3 is that it provides a test for modularity which only involves the min path sets with no mention of the min cuts. The utility of this will become more apparent in later proofs. We give the usual definition of modules for completeness here as

<u>Proposition 4</u>.

Let (C,ϕ) be a coherent system with all components essential and A be a nonempty subset of C. A is a module of $(C,\phi) \Leftrightarrow \phi(X) = \psi(\Gamma(X|_A), X|_{C-A})$, where:

$X|_A$ is X restricted to A

(A,Γ) is a coherent system

$(\{c_A\} \cup (C-A), \psi)$ is a coherent system, and

$(\Gamma(X|_A), X|_{C-A})(c) = X(c)$ for $c\ \varepsilon\ C - A$, $= \Gamma(X|_A)$ for $c = c_A$.

We remark that if A is a module, then Γ and ψ in Proposition 4 are uniquely determined. We can therefore refer unambiguously to the coherent system (A,Γ) as a module of (C,ϕ) whenever A is a module of (C,ϕ). In fact, (A,Γ) has min path sets $\mathcal{P}|_A$ and min cut sets $\mathcal{K}|_A$. See [ ] for a proof that this characterization is equivalent to (9) and hence to our definition.

It is natural to conjecture that if $\mathcal{P}|_A$ and $\mathcal{K}|_A$ are candidates for min paths and min cuts on A, then A is a module. That is, suppose $\mathcal{P}|_A$ in place of $\mathcal{P}$ satisfies (3) and also $\mathcal{K}|_A$ in place of $\mathcal{P}$ satisfies (3). Then there exists a coherent system (A,Γ) with min path sets $\mathcal{P}|_A$ and a coherent system (A,Γ^*) with min cut sets $\mathcal{K}|_A$. The conjecture that A is then a module, i.e., that $\Gamma = \Gamma^*$, is false. For example, take $C = \{1,2,3,4,5\}$, $\mathcal{P} = \{\{1,3,5\},\{2,3,4\},\{2,5\},\{1,4\}\}$, $\mathcal{K} = \{\{1,3,5\},\{2,3,4\},\{1,2\},\{4,5\}\}$ and $A = \{1,2,4,5\}$. Verification of the example is left to the interested reader.

3.2 Preliminary Properties of Modules.

Intuitively, one expects that a module of a module is again a module. This is true and is obvious from our definition since if $A \supseteq B$, then $(\mathcal{F}|_A)|_B = \mathcal{F}|_B$, for any family $\mathcal{F}$ of subsets.

It is easy to see that (A,Γ) is a module of (C,ϕ) if and only if (A,Γ^d) is a module of (C,ϕ^d). Using this, one can give a dual result for each of our results in which paths are replaced by cuts. In particular, Proposition 3 holds when in (9) and (10), paths are replaced by cuts.

The following properties are regarded as lemmas because of their use in proving the three modules theorem. We will use equation (9) or (10) as equivalent to modularity without further mention.

<u>Two Modules Lemma</u>:

Let A be a nonempty proper subset of C and (C,ϕ) be a coherent system all of whose components are essential, with min path sets $\mathcal{P}$ and min cut sets $\mathcal{K}$. Then A and $C - A$ are both modules if and only if

$$\text{either (a)} \quad \forall\, P \in \mathcal{P},\ P \subseteq A \ \text{ or } \ P \subseteq C - A$$

$$\text{or (b)} \quad \forall\, K \in \mathcal{K},\ K \subseteq A \ \text{ or } \ K \subseteq C - A$$

<u>Proof</u>:

Suppose A and $C - A$ are modules and suppose (a) fails. Then $\exists\, P^* \in \mathcal{P} \ni P^*A \neq \emptyset$ and $P^*(C-A) \neq \emptyset$. First we show that $\forall\, P \in \mathcal{P}$, $PA \neq \emptyset$ and $P(C-A) \neq \emptyset$. Suppose not. If $P \subseteq A \Rightarrow P^*A \cup P(C-A) = P^*A$ is a path, a contradiction since $P^*A \subset P^*$. Analogous reasoning applies if $P \subseteq C - A$ hence every min path intersects both A and $C - A$. Now, if in addition (b) were to fail, then $\exists\, K^* \in \mathcal{K} \ni K^*A \neq \emptyset$ and $K^*(C-A) \neq \emptyset$. By modularity of A, $\forall\, P \in \mathcal{P}$, we have $P(K^*A) \neq \emptyset$. However since $K^*A \subset K^*$, this contradicts the supposed minimality of K^*. The converse follows easily.

If (a) holds, we say A and $C - A$ are in parallel, while if (b) holds, we say A and $C - A$ are in series. The parallel and series designation is not arbitrary; indeed if (A,Γ') and $(C-A,\Gamma'')$ are modules of (C,ϕ), then

$$\phi(X) = \text{Max}\,\{\Gamma'(X|_A),\Gamma''(X|_{C-A})\}$$

or

$$= \text{Min}\,\{\Gamma'(X|_A),\Gamma''(X|_{C-A})\}$$

according as (a) or (b) holds respectively.

<u>Intersection Lemma</u>:

Let A and B be modules of (C,ϕ), a coherent system. Then $AB = \emptyset$ or AB is a module of (C,ϕ).

<u>Proof</u>:

Suppose $AB \neq \emptyset$. If P and Q are min paths which intersect AB, then $PA \cup Q(C-A)$ is a min path which intersects B, so $(PA \cup Q(C-A))B \cup Q(C-B) = PAB \cup Q(C-AB)$ is a min path, showing AB is a module.

<u>Difference Lemma</u>:

Let A and B be modules of (C,ϕ), a coherent system with all components essential. If $A - B$ and $B - A$ are both nonempty, then both are modules.

<u>Proof</u>:

Let $\mathcal{P}$ denote the min path sets and $\mathcal{K}$ denote the min cut sets. We simply show $A - B$ must be a module. By the two modules lemma, it is sufficient to show that

$$\begin{array}{lll} \text{either} & \text{(a)} & \forall\ PA \in \mathcal{P}|_A,\ PA \subseteq A - B \text{ or } PA \subseteq AB \\ \text{or} & \text{(b)} & \forall\ KA \in \mathcal{K}|_A,\ KA \subseteq A - B \text{ or } KA \subseteq AB \end{array}$$

If (a) fails, then $\exists\ P^*A \in \mathcal{P}|_A \ni P^*(A-B) \neq \emptyset$ and $P^*AB \neq \emptyset$. By hypothesis, $\exists\ Q^* \in \mathcal{P} \ni Q^*(B-A) \neq \emptyset$. It must be that $Q^*AB \neq \emptyset$ or else, using modularity of B, it would follow that P^* is not minimal.

Similarly, minimality of Q^* implies $\forall$ $PA \in P|_A$, $P(A-B) \neq \emptyset$ and $PAB \neq \emptyset$. Now suppose (b) fails so $\exists$ $K^*A \in K|_A \ni K^*(A-B) \neq \emptyset$ and $K^*AB \neq \emptyset$. It follows, by modularity of AB, that K^*AB intersects every element of $P|_A$, hence K^*AB is a cut set of module A. This contradicts the minimality of K^*A and the proof is complete.

3.3 <u>Three Modules Theorem</u>.

A result of significant importance to the understanding of modules is the three modules theorem. It was proved in [1] in the more general setting of switching functions. A much simplified proof appears in [4] for coherent systems, using a Boolean function approach. Our proof, also much simplified, shows this paper's methods to be as fundamental as the Boolean function approach.

The following theorem and the two modules lemma completely specify how two modules can coexist in a coherent system.

<u>Three Modules Theorem</u>:

Let (C,ϕ) be any coherent system with all components essential. If A and B are modules such that $A - B$, AB and $B - A$ are nonempty, then

(i) $A - B$, AB and $B - A$ are modules.

(ii) $A \Delta B$ and $A \cup B$ are modules.

(iii) The three modules $A - B$, AB and $B - A$ appear in either parallel or series.

Proof:

Let $\mathcal{P}$ denote the min path sets and $\mathcal{K}$ denote the min cut sets of (C,ϕ). The intersection and difference lemmas prove (i). Let E denote $A \cup B$. As for (ii), repeated application of the two modules lemma and the fact that every min path must intersect every min cut in a coherent system imply that

$$\text{either (a) } \forall\, PE \in \mathcal{P}|_E,\quad PE \subseteq A-B \text{ or } PE \subseteq AB \text{ or } PE \subseteq B-A$$
$$\text{or (b) } \forall\, KE \in \mathcal{K}|_E,\quad KE \subseteq A-B \text{ or } KE \subseteq AB \text{ or } KE \subseteq B-A.$$

Without loss of generality, we will assume (b) holds. Again from the two modules lemma, it follows that if (b) holds then

$$\text{(c) } \forall\, PE \in \mathcal{P}|_E,\quad P(A-B) \neq \emptyset \text{ and } PAB \neq \emptyset \text{ and } P(B-A) \neq \emptyset.$$

Now the modularity of E can be verified easily. Let $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ both intersect E. They both intersect B so $PB \cup Q(C-B) \in \mathcal{P}$. But this min path and P both intersect A so $PA \cup P(B-A) \cup Q(C-E) = PE \cup Q(C-E) \in \mathcal{P}$, showing $E = A \cup B$ is a module.

The modularity of $A \cup B$, the fact either (a) or (b) holds and the two modules lemma establish that the symmetric difference $A \Delta B = (A-B) \cup (B-A)$ is a module.

As for (iii), if (a) holds then $A - B$ and B are in parallel as are B and $B - A$ hence all three appear in parallel. A similar argument applies if (b) holds, showing the three modules to be in series. This completes the proof.

3.4 An Application.

An excellent application of the three modules theorem concerns "maximal" modules. We give the results here and refer the interested reader to [4] for the proof.

We will say a module $M \neq C$ of (C,ϕ) is *maximal* if it is set maximal with respect to being a module other than C.

Let $\mathcal{M}$ be the set of maximal modules of (C,ϕ), a coherent system with all components essential. Then

either (1) $\mathcal{M}$ is a partition of C

or (2) $\bar{\mathcal{M}} \triangleq \{C - M \mid M \in \mathcal{M}\}$ is a set of modules which partition C and which appear in either series or parallel.

This application is discussed in [3] while in [2] the author gives an algorithm for finding the maximal modules.

Acknowledgment.

The author wishes to credit E. Lawler with first pointing out the similarity of blocking systems and coherent systems to him. Also, the author is indebted to J. Esary for his helpful suggestions and to L. Shapley for bringing [2], [3] and [10] to his attention.

REFERENCES

[1] R. L. Ashenhurst, "The Decomposition of Switching Functions," Proceedings of an International Symposium on the Theory of Switching, Part I (Vol XXIX, Ann. Computation Lab. Harvard), Harvard University Press, Cambridge, 1959, pp. 75-116.

[2] L. J. Billera, "On the Composition and Decomposition of Clutters," *Jour. Combinatorial Theory.*

[3] L. J. Billera, "Clutter Decomposition and Monotonic Boolean Functions," *Annals of the New York Academy of Sciences*, Vol. 175, Article 1, pp. 41-48 (1970).

[4] Z. W. Birnbaum and J. D. Esary, "Modules of Coherent Binary Systems," *Jour. Soc. Industrial Applied Mathematics*, Vol. 13, No. 2, pp. 444-462 (June 1965).

[5] Z. W. Birnbaum, J. D. Esary and A. W. Marshall, "A Stochastic Characterization of Wear-Out for Components and Systems," *Annals of Mathematical Statistics*, Vol. 37, No. 4, pp. 816-825 (August 1966).

[6] Z. W. Birnbaum, J. D. Esary and S. C. Saunders, "Multi-Component Systems and Structures and Their Reliability," *Technometrics*, Vol. 3, No. 1, pp. 55-77 (February 1961).

[7] Jack Edmonds and D. R. Fulkerson, "Bottleneck Extrema," RAND Report RM-5375-PR (January 1968).

[8] J. D. Esary and F. Proschan, "Coherent Structures of Non-Identical Components," *Technometrics*, Vol. 5, No. 2, pp. 191-209 (May 1963).

[9] D. R. Fulkerson, Networks, Frames, Blocking Systems," Mathematics of the Decision Sciences, Part I, Vol. II, pp. 303-334.

[10] L. S. Shapley, "Compound Simple Games, III: On Committees," The RAND Corporation RM-5438-PR, October 1967.

INITIAL DISTRIBUTION LIST

Defense Documentation Center (DDC) Cameron Station Alexandria, Virginia 22314	20
Dean C. E. Menneken Dean of Research Administration Naval Postgraduate School Monterey, California 93940	1
Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
Library, Code 55 Department of Operations Research and Administrative Sciences Naval Postgraduate School Monterey, California 93940	2
James D. Esary, Code 55Ey Department of Operations Research and Administrative Sciences Naval Postgraduate School Monterey, California 93940	1
Gilbert T. Howard, Code 55Hk Department of Operations Research and Administrative Sciences Naval Postgraduate School Monterey, California 93940	1
E. Lawler Department of Electrical Engineering University of Michigan Ann Arbor, Michigan 48104	1
R. Barlow Department of Industrial Engineering and Operations Research University of California Berkeley, California 94720	1
Dr. William Mitchell Assistant Professor California State College Hayward, California 94542	1

Richard W. Butterworth, Code 55Bd 10
Department of Operations Research
and Administrative Sciences
Naval Postgraduate School
Monterey, California 93940

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*

Naval Postgraduate School
Monterey, California 93940

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

3. REPORT TITLE

A Set Theoretic Treatment of Coherent Systems

4. DESCRIPTIVE NOTES *(Type of report and, inclusive dates)*

Technical Report - June 1971

5. AUTHOR(S) *(First name, middle initial, last name)*

Richard W. Butterworth

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1971	24	10

8a. CONTRACT OR GRANT NO.

b. PROJECT NO.

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

This paper develops properties of coherent systems from a set theoretic viewpoint with particular emphasis on modules of coherent systems. The methods used here demonstrate concisely the previously reported properties of modules as well as some new properties. Specifically, min path sets and min cut sets of a coherent system are given several characterizations. The modularity of a subset of components is given a new characterization which we use as a definition. The situation under which a set and its complement are simultaneously modules is characterized. Finally, the established three modules theorem is given a new proof.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Coherent Systems						
Modules						
Blocking Systems						
Simple Games						